

DEFORMATION OF QUINTIC THREEFOLDS TO THE CHORDAL VARIETY

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ABSTRACT. We consider a family of quintic threefolds specializing to a certain reducible threefold. What are the flat limits of the rational curves in this degeneration? A first approximation to the answer is the description of the space of genus zero stable morphisms to the central fiber (as defined by J. Li). As an elementary application of the analysis, we prove the existence of rigid stable maps of arbitrary genus and sufficiently high degree to very general quintics.

1. INTRODUCTION

1.1. Algebraic Curves on Calabi-Yau Threefolds. The study of algebraic curves on Calabi-Yau manifolds is a major topic in algebraic and enumerative geometry, partly motivated by its connections with physics. From the point of view of classical algebraic geometry, it is an unfortunate and surprising state of affairs that these objects appear to be largely out of reach, despite the deep understanding of their enumerative behavior provided by Gromov-Witten theory.

Part of the reason is that curves on Calabi-Yau manifolds are, in a sense, extremely rigid. One illustration is the fact that the "expected" dimension of the space of algebraic curves of any genus and homology class on a Calabi-Yau threefold is always zero. A longstanding classical open question motivated by this observation is the Clemens conjecture. In its minimal form, the conjecture states that a very general quintic threefold contains finitely many rational curves of each degree. The conjecture has been proved in degrees up to 11 in a series of papers by Katz, Johnsen-Kleimann and Cotterill [Ka86a, JK96, Co05, Co12]. However, very little is known in large degrees. Clemens proved the existence of smooth infinitesimally rigid rational curves of arbitrarily large degree as an ingredient in his theorem on algebraic versus homological equivalence of 1-cycles on quintic threefolds and later Katz refined this construction to prove that such rigid curves occur in all degrees.

A folklore approach to the problem is by specialization. Very naively, if one could exhibit a sufficiently general family of quintics specializing to a reducible variety and locate the flat limits of the curves on the degenerate variety, finiteness would obviously follow. Recently, there has been a great deal of renewed interest in a related idea, from the point of view of enumerative geometry. Toric degenerations of Calabi-Yau threefolds have been proposed as a means to understanding mirror symmetry, in what is known as the Gross-Siebert program [GS06, GS10, GS09, Bu10].

A simple but already very interesting example is the degeneration of a family quintics to the simplex of coordinate hyperplanes, i.e. the reducible hypersurface $Z_0 Z_1 Z_2 Z_3 Z_4 = 0$. The project of describing explicitly the limits of the rational curves from the general fibers has been carried out in degree $d = 1$, cf. [Ka86b, Ni15]. However, extending this analysis to arbitrary degree seems extremely difficult.

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The main purpose of this paper is to exhibit an alternative degeneration of quintic threefolds which provides a totally different type of leverage (i.e. it is not toric) in the study of limits of rational curves and to show some definite progress towards this goal.

1.2. The Approach by Specialization. The degeneration we will look at is obtained by running semistable reduction on a very general pencil of quintic threefolds specializing to the secant variety of a normal elliptic curve in the ambient projective space. Let $E \subset \mathbb{P}^4$ be a smooth nondegenerate² curve of genus 1 and degree 5 and let N be the chordal variety of E , that is, the singular threefold swept out by all lines connecting pairs of point on E . Formally, consider the diagram

$$\begin{array}{ccccc} \tilde{N} & \longrightarrow & U & \longrightarrow & \mathbb{P}^4 \\ \downarrow & & \downarrow & & \\ E^{(2)} & \longrightarrow & \mathbb{G}(1, 4) & & \end{array}$$

where $E^{(k)}$ is the symmetric k th power of E or equivalently the Hilbert scheme of k points on E , $\mathbb{G}(1, 4)$ is the Grassmannian of lines in \mathbb{P}^4 and $U \rightarrow \mathbb{G}(1, 4)$ is the universal family of lines. The map $E^{(2)} \rightarrow \mathbb{G}(1, 4)$ is defined by sending a length two subscheme of E to the unique line in \mathbb{P}^4 which contains it. Define $\tilde{N} = E^{(2)} \times_{\mathbb{G}(1, 4)} U$. It can be proved that $\deg N = 5$. We'll briefly review this calculation and other basic geometric properties at the beginning of Section 2.

Consider a very general pencil of quintic threefolds whose fiber over $0 \in \mathbb{A}^1 \subset \mathbb{P}^1$ is the secant variety N . The total space of the pencil is the variety $P_N + tP = 0$ in $\mathbb{A}_t^1 \times \mathbb{P}^4$ where P_N is the homogeneous polynomial of N and P is a degree 5 homogenous element of $\mathbb{C}[Z_0, Z_1, Z_2, Z_3, Z_4]$. Seen over \mathbb{A}_t^1 , this is a family of quintics degenerating to N . The main observation in this paper is that running semistable reduction on this family yields a central fiber with two (smooth and transverse) components both admitting nonconstant maps to a smooth algebraic curve of genus one.

Section 2 is devoted to the semistable reduction calculation. There are two steps: (1) make a base change of order 3 totally ramified at $0 \in \mathbb{A}^1$; and (2) blow up the curve $E \times \{0\}$ in the total space of the family. Said differently, we are looking at the proper transform X of $Z = V(P_N + t^3P)$ in the blowup of $\mathbb{A}^1 \times \mathbb{P}^4$ along $\{0\} \times E$. It will turn out that X is smooth and its central fiber X_0 has two normal crossing smooth components $Y_1 = \tilde{N}$, which is fibered over $\text{Pic}^2 E$ and the exceptional divisor Y_2 , which clearly admits a map to E . The fibers of $Y_1 \rightarrow \text{Pic}^2 E$ are abstractly \mathbb{F}^1 's, while the fibers of $Y_2 \rightarrow E$ are (smooth with 25 exceptions) special cubic surfaces, namely cubic surfaces which can be expressed as a triple cover of \mathbb{P}^2 totally ramified along a plane cubic curve. The 25 singular fibers, occurring at points which we will denote $p_1, p_2, \dots, p_{25} \in E$, are projective cones over a smooth plane cubic curve. The intersection $Y_1 \cap Y_2$ is isomorphic to $E \times E$ and the two maps above restrict on the intersection to the tensor product map $\otimes : E \times E \rightarrow \text{Pic}^2 E$ and projection to one of the factors respectively.

It may be useful to abstract away the essential features of the configuration above. What we have is a family $\pi : X \rightarrow \Delta = \mathbb{A}^1$ with smooth total space such that $X_0 = Y_1 \cup Y_2$

²In this context, nondegenerate means it is not contained in any hyperplane.

with $Y_1 \cap Y_2 = E_1 \times E_2$ and $E_1 \cong E_2$ isomorphic smooth genus 1 curves, Y_i smooth and admitting a map $\varphi_i : Y_i \rightarrow E_i$ such that the diagram

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{\quad} & E_i \\ \downarrow & \nearrow & \\ Y_i & & \end{array}$$

in which the vertical arrow is the closed embedding, the diagonal arrow is φ_i and the horizontal arrow is the projection to the respective factor, is commutative for $i = 1, 2$. Moreover, the fibers of the pair $(Y_i, E_1 \times E_2)$ over E_i are generically log K3 pairs, i.e. pairs consisting of a del Pezzo surface and a smooth anti canonical divisor.

The (purely aesthetic) change of perspective from $E \times E$ to $E_1 \times E_2$ is obtained by considering the isomorphism $(\otimes, \text{proj}_1) : E \times E \rightarrow E_1 \times E_2$, where $E_1 = \text{Pic}^2 E$ and $E_2 = E$. A choice of one of the 25 hyperflexes of $E \subset \mathbb{P}^4$ gives a convenient identification of $E_1 = \text{Pic}^2 E$ with E . If h is the hyperflex, the isomorphism is the map $p \mapsto \mathcal{O}_E(p + h)$.

Remark 1.1. In this paper we will only be concerned with such a family of quintic threefolds, but it seems very likely cf. [GP00] that degenerations fitting into the pattern above also exist for other deformation classes of Calabi-Yau threefolds.

1.3. Limits of Rational Curves. As we have explained in the previous section, the purpose of the construction above is to analyze the reducible curves in the central fiber X_0 which are limits of rational curves from nearby fibers X_t , $t \neq 0$. The fact that both components admit maps to genus one curves confines all the irreducible components of any limit to the fibers of these maps. There is some ambiguity about what we mean by limit: for now we will think of the limits as being the 1-cycles, while in the rest of the paper we will use the theory of relative stable maps. Consider the map

$$\left\{ \begin{array}{l} \text{1-cycles on } X_0 \text{ which are limits} \\ \text{of rational curves from nearby fibers} \end{array} \right\} \xrightarrow{\quad \Phi \quad} \left\{ \begin{array}{l} \text{finite collections of} \\ \text{closed points on } E \end{array} \right\}$$

which associates to each 1-cycle $L = a_1 D_1 + \dots + a_n D_n \in Z^1(X_0)$ which is a limit of rational curves the unordered collection of all points $p \in E_1$ and $p \in E_2$ such that $\varphi_i^{-1}(p)$ contains some irreducible component D_j . Let Φ_d be the restriction of Φ to the subset of limits of degree d curves.

It is easy to prove that Φ_d is finite-to-one. Therefore, the set of limits is finite (which is obviously equivalent to the Clemens conjecture being true) if and only if $\text{Im}(\Phi_d)$ is finite. It appears that $\text{Im}(\Phi_d)$ is governed by complicated but purely combinatorial patterns related to the group structure of E . This speculation is illustrated by the following (slightly pretentious) unresolved question, which can be proved for $d \leq 5$ using the methods presented here.

Question 1.2. *Is it necessarily true that all entries of any element of $\text{Im}(\Phi)$ are rational linear combinations³ of the 25 intersection points of the original normal elliptic curve E with the base locus of the pencil?*

³By this we mean $lq \sim k_1 p_1 + \dots + k_{25} p_{25}$ for $l, k_1, \dots, k_{25} \in \mathbb{Z}$, $l \neq 0$.

The main goal of this paper is to describe the spaces to genus zero stable maps to X_0 (or rather \mathfrak{X}_0), as defined by Li [Li01, Li02]. We will obtain a manageable description of these spaces for any degree d , involving certain spaces of rational curves on del Pezzo surfaces and some combinatorics involving discrete differential equations for torus-valued functions on a graph.

Moreover, we will encounter a very striking phenomenon: the families of maps sweeping out a locus of dimension two or more on X_0 have, in a certain sense, contribution 0 to the virtual count. Unfortunately, there are some serious technical obstacles to making sense precisely of this remark, but this seems at least to warrant further investigation.

Finally, we would like to briefly discuss the $g > 0$ case. The analysis above cannot be totally generalized to this case, but it can be for degenerate stable maps whose dual graph has $h^1 = g$, since this is a purely topological condition which forces all components to be rational. Using this, we may extend (a slightly weaker form of) Katz' result about the existence of rigid smooth curves to arbitrary genus. We remark that the old argument involving deformation of curves from quartic surfaces cannot be extended to $g > 0$.

Theorem 1.3. *Let $g \geq 0$, $d \gg g$ and X a general quintic threefold. Then X admits a rigid⁴ degree d stable map $f : C \rightarrow X$ whose source C is a smooth curve of genus g .*

Remark 1.4. The possibility of f factoring through an unramified cover of smooth curves with source C is not ruled out in general, but this situation can only occur for special d and g .

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2. THE SEIMSTABLE REDUCTION CALCULATION

2.1. Secant Variety of a Normal Elliptic Curve. We begin with an elementary review of the secant variety of a normal elliptic curve in \mathbb{P}^4 . We use the same notation as above: E is the normal elliptic curve and N is the secant variety, which by definition is the singular threefold swept out by all the lines intersecting E in a scheme of length 2. Let \tilde{P} be the blow up of \mathbb{P}^4 along E . There is an obvious tower of \mathbb{P}^1 -bundles $\tilde{N} \rightarrow \text{Sym}^2 E \rightarrow \text{Pic}^2 E$ and thus \tilde{N} is smooth. It follows that for any $D \in \text{Pic}^2 E$, the fiber \tilde{N}_D is some rational ruled surface \mathbb{F}_n .

Lemma 2.1. The degree of N in \mathbb{P}^4 is equal to 5.

⁴By rigid, we mean that $[f]$ is an isolated point of $\overline{M}_g(X, d)$. Ideally, this should be strengthened to infinitesimal rigidity, i.e. $\text{Def}^1(f) = 0$, especially since the methods I used seem to allow it.

Sketch of proof. We may project away from a general line $l \subset \mathbb{P}^4$ onto \mathbb{P}^2 and note that intersection points of l with N correspond to the nodes of the projection of E . Since the projection of E to \mathbb{P}^2 has degree 5 and geometric genus 1, the genus-degree formula implies that the number of nodes is equal to 5. \square

Lemma 2.2. The map $j : \tilde{N} \rightarrow \tilde{P}$ is a closed embedding.

Sketch of Proof. It suffices to show that j is injective and $dj : T\tilde{N} \rightarrow j^*T\tilde{P}$ is fiberwise injective. For the first claim, assume that there exist two distinct points $p, q \in \tilde{N}$ such that $j(p) = j(q)$. Let $\tilde{l}_p, \tilde{l}_q \subset \tilde{N}$ be the two secant lines containing p and q and let l_p, l_q be their images in \mathbb{P}^4 . Clearly, $l_p \neq l_q$. Assume first that $l_p \cap l_q \notin E$. Then the length 4 subscheme $D_{p,q} = E \cap (l_p \cup l_q)$ of E is contained in the projective subspace of \mathbb{P}^4 spanned by l_p and l_q , which has dimension 2. However, by Riemann-Roch,

$$h^0(\mathcal{O}_E(1) \otimes \mathcal{O}_E(-D_{p,q})) = 1,$$

leading to contradiction. If $l_p \cap l_q \in E$, then $\tilde{l}_p \cap \tilde{l}_q = \emptyset$, which is also impossible.

The second claim is an infinitesimal version of the first. Assume that there is a tangent vector of to \tilde{N} which is killed by dj . Then there exists a line and a first order thickening $\tilde{l} \subset \hat{l} \subset \tilde{N}$ such that the corresponding section of $N_{\tilde{l}/\tilde{P}}$ vanishes at some point, or equivalently, l' is contained in a plane, where $l \subset l'$ is the image in \mathbb{P}^4 of $\tilde{l} \subset \hat{l}$. Then an argument similar to the one above leads to contradiction. \square

Claim 2.3. All the fibers of $\tilde{N} \rightarrow \text{Pic}^2 E$ are isomorphic to \mathbb{F}_1 .

Proof. One way to prove this is via an indirect argument starting with the Hirzebruch surface. Let $f, e_\infty, e_0 \in \text{Pic } \mathbb{F}_1$ be the classes of the fibration, of the directrix and of the preimage of a general line from the projective plane obtained by blowing down the directrix respectively. These classes generate $\text{Pic } \mathbb{F}_1$, subject to the relation $e_0 = e_\infty + f$. The line bundle $\mathcal{O}_{\mathbb{F}_1}(e_\infty + 2f)$ defines the standard embedding of \mathbb{F}_1 into \mathbb{P}^4 .

Consider the anticanonical class $-K_{\mathbb{F}_1} = \mathcal{O}_{\mathbb{F}_1}(2e_\infty + 3f)$. The divisors in the associated linear system are curves of arithmetic genus one, intersecting the directrix once and the fibers of the ruling twice. It is not hard to see that we can find such a divisor, which is abstractly isomorphic to our curve E and such that the divisor class on E cut by the ruling is precisely D . Then embedding the pair $E \subset \mathbb{F}_1$ in \mathbb{P}^4 , it is not hard to see that E is an elliptic normal curve and that the \mathbb{F}_1 surface is precisely the surface swept out by the lines connecting pairs of points on E whose sum as divisor classes is precisely D . Since the action of $\text{PGL}(5, \mathbb{C})$ on the set of normal elliptic curves of a fixed j -invariant is transitive, this argument proves the claim that \tilde{N}_D is an \mathbb{F}_1 surface. \square

Alternative Proof. The claim also follows from the observation that \tilde{N}_D is a *minimal degree variety*, due to the classification proved in [EH87]. \square

Note that the intersection S of \tilde{N} with the exceptional divisor T of \tilde{P} is canonically isomorphic to $E \times E$. Roughly, any point in the intersection is specified by choosing a point on E and an infinitesimal direction inside N from the chosen point, the latter also being specified by a point on E . The restriction of $\tilde{N} \rightarrow \text{Pic}^2 E$ to this locus is the tensor product map μ by construction. In this section, we order the factors of $S = E \times E$ such

that the restriction $S \rightarrow E$ of the blowdown map is projection to the first factor. We will usually identify $\mu^{-1}(D)$ with E in this way.

Claim 2.4. (a) The point where the directrix of \tilde{N}_D intersects $\mu^{-1}(D) \cong E$ is the one whose divisor class is $\mathcal{O}_E(1) \otimes \mathcal{O}_E(-2D)$.

(b) The restriction of the hyperplane class of T_p to $S_p \cong E$ is $\mathcal{O}_E(1) \otimes \mathcal{O}_E(-2p)$, for all closed $p \in E$.

Proof. Part (a) follows from the proof of 2.1. For part (b), simply pick a hyperplane containing the line tangent to $E \subset \mathbb{P}^4$ at p and project away from the tangent line. Note that S_p sits inside $T_p \cong \mathbb{P}^2$ as a smooth plane cubic. \square

Denote the blowdown map $\tilde{P} \rightarrow \mathbb{P}^4$ by ρ and its restriction to T by $\tau : T \rightarrow E$. Of course, T is the projectivization of the normal bundle of E , i.e. $T = \underline{\text{Proj}} \text{Sym } \mathcal{N}_{E/\mathbb{P}^4}^\vee$ and as such comes equipped with a tautological line bundle $\mathcal{O}_T(-1)$ which is isomorphic to $\mathcal{O}_{\tilde{P}}(T) \otimes \mathcal{O}_T$.

Calculation 2.5. The following two isomorphisms hold

$$\rho^* \mathcal{O}_{\mathbb{P}^4}(N) \cong \mathcal{O}_{\tilde{P}}(\tilde{N} + 3T) \quad (2.1.a)$$

$$\mathcal{O}_T(S) \cong \mathcal{O}_T(3) \otimes \tau^* \mathcal{O}_E(5) \quad (2.1.b)$$

Proof. We must have $\rho^* N = \tilde{N} + kT$ for some k as Cartier divisors. For a line L contained inside the fibers of τ we have $(\rho^* N \cdot L) = 0$, $(\tilde{N} \cdot L) = 3$ since $S_p \subset T_p$ has degree 3 and $(T \cdot L) = -1$, hence $k = 3$. For the second part, tensor both sides with \mathcal{O}_T . The left hand side and the right hand side become respectively

$$\rho^* \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{O}_T \cong \tau^* \mathcal{O}_E(5) \text{ and } \mathcal{O}_{\tilde{P}}(\tilde{N} + 3T) \otimes \mathcal{O}_T \cong \mathcal{O}_T(-3) \otimes \mathcal{O}_T(S),$$

proving (2.1.b). \square

Lemma 2.6. The ideal sheaf $\mathcal{I}_{N/\mathbb{P}^4}$, viewed as a subsheaf of the structure sheaf of \mathbb{P}^4 , is contained inside $\mathcal{I}_{E/\mathbb{P}^4}^3$.

Proof. Is it clear from (2.1.a) that there exists a natural homomorphism of $\mathcal{O}_{\tilde{P}}$ -modules $\mathcal{O}_{\tilde{P}}(3T) \rightarrow \rho^* \mathcal{O}_{\mathbb{P}^4}(N)$, dual to some homomorphism $\rho^* \mathcal{I}_{N/\mathbb{P}^4} \rightarrow \mathcal{I}_{T/\tilde{P}}^3$. By the adjoint property, we get a map $\mathcal{I}_{N/\mathbb{P}^4} \rightarrow \rho_* \mathcal{I}_{T/\tilde{P}}^3$ and it is not hard to check that the latter sheaf is simply $\mathcal{I}_{E/\mathbb{P}^4}^3$. The verification of the fact that the triangle formed by the $\mathcal{O}_{\mathbb{P}^4}$ -module homomorphism defined above and the inclusion maps into the structure sheaf commutes is skipped. \square

Remark 2.7. Given the discussion of the specialization of quintics to the the simplex of coordinate hyperplanes in 1.1, it's hard to resist the temptation of making the observation that this is in fact a flat limit of a family of such secant varieties when E degenerates to a 5-cycle of lines. Indeed, let $L_{i,j}$ be the line of equation $Z_\alpha = Z_\beta = Z_\gamma = 0$ in \mathbb{P}^4 where $\{i, j, \alpha, \beta, \gamma\} = \{0, 1, 2, 3, 4\}$. Then $E^\dagger = L_{0,1} \cup L_{1,2} \cup L_{2,3} \cup L_{3,4} \cup L_{0,4}$ is a reducible curve of degree 5 and arithmetic genus 1 whose secant variety is clearly the 5-simplex of coordinate hyperplanes $Z_0 Z_1 Z_2 Z_3 Z_4 = 0$. For instance, the lines secant to both $L_{0,1}$ and $L_{2,3}$ swipe out the $Z_4 = 0$ hyperplane; similar reasoning applies to the other coordinate hyperplanes.

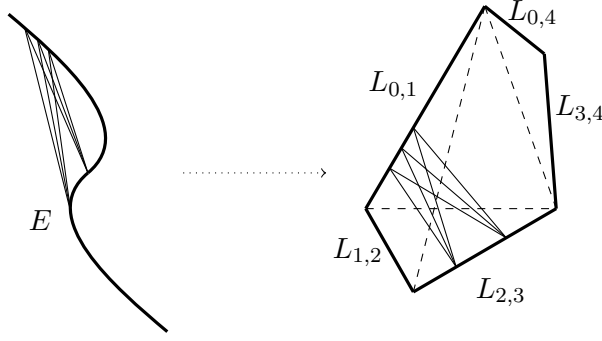


Fig. 2.1. N specializing to $Z_0Z_1Z_2Z_3Z_4 = 0$.

It is not hard to check that E^\dagger is smoothable, proving that secant varieties indeed specialize to unions of five hyperplanes.

2.2. Base Change and Blowup. The previous lemma suggests that in order to obtain a good central fiber, it is necessary to perform a base change of order 3 before blowing up the singular locus. Set $\mathcal{N} = \mathcal{N}_{E \times \{0\}/\mathbb{P}^4}$. The normal bundle of $E \times \{0\}$ in $\mathbb{P}^4 \times \mathbb{A}^1$ is isomorphic to $\mathcal{N} \oplus \mathcal{O}_E$. To be pedantic now rather than confusing later, we should clarify that the summand \mathcal{O}_E should actually be written as $\mathcal{O}_E \otimes T_0\mathbb{A}^1 = \mathcal{O}_E \otimes \mathbb{C}\langle t \rangle^\vee$, where t is the affine coordinate of \mathbb{A}^1 .

Let W be the blowup of $\mathbb{P}^4 \times \mathbb{A}^1$ along $E \times \{0\}$. From now on we will usually write just E instead of $E \times \{0\}$. The central fiber W_0 has two components: $W_{0,1} \cong \tilde{P}$ and

$$W_{0,2} = \underline{\text{Proj}} \text{ Sym } (\mathcal{N} \oplus \mathcal{O}_E)^\vee.$$

Note that the inclusion $\mathcal{O}_E \subset \mathcal{O}_E \oplus \mathcal{N}$ induces a section $\sigma : E \rightarrow W_{0,2}$ whose image is geometrically the intersection of the proper transform of $E \times \mathbb{A}^1$ with the central fiber. Dually, it defines a distinguished section $t \in H^0(\mathcal{O}_{W_{0,2}}(T))$ which we may identify with the affine coordinate above.

After a base change of order 3, a pencil of quintics specializing to N has total space $Z = V(P_N + t^3P)$ where we choose P to be a very general homogeneous quintic polynomial and P_N is the homogeneous equation of N . The central fiber X_0 of the proper transform X of Z correspondingly has two components: $Y_1 = \tilde{N} \subset W_{0,1}$ and $Y_2 \subset W_{0,2}$. There seems to be little choice in describing Y_2 besides writing explicit equations. We may start from the definition of Y_2

$$Y_2 = \underline{\text{Proj}} \bigoplus_{k \geq 0} \mathcal{I}_{E/Z}^k / \mathcal{I}_{E/Z}^{k+1} \quad (2.2)$$

but it is hard to extract all the geometric information we need directly from it. We will show that Y_2 is the total space of a family of cubic surfaces in the fibers of the projective bundle $\varphi_2 : W_{0,2} \rightarrow E$.

Note that Lemma 2.6 further implies that $\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1} \subset \mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^3$. Indeed, given the shape of the equation defining Z , we have

$$\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1} \subseteq \mathcal{I}_{N/\mathbb{P}^4} \boxtimes \mathcal{O}_{\mathbb{A}^1} + \mathcal{I}_{\mathbb{P}^4 \times \{0\}/\mathbb{P}^4 \times \mathbb{A}^1}^3 \subseteq \mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^3.$$

Consider the following commutative diagram of coherent sheaves on E

$$\begin{array}{ccccc}
\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1} \otimes \mathcal{O}_E & \longrightarrow & \mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^3 \otimes \mathcal{O}_E & \longrightarrow & \mathrm{Sym}^3(\mathcal{N} \oplus \mathcal{O}_E)^\vee \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_{N/\mathbb{P}^4} \otimes \mathcal{O}_E & \longrightarrow & \mathcal{I}_{E/\mathbb{P}^4}^3 \otimes \mathcal{O}_E & \longrightarrow & \mathrm{Sym}^3 \mathcal{N}^\vee
\end{array}$$

The top row is related to the inclusion $Y_2 \subset W_{0,2}$, while the bottom row is related to the inclusion $S \subset T$. The left vertical arrow is an isomorphism as it is simply the restriction to E of the isomorphism $\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1}|_{(t=0)} \cong \mathcal{I}_{N/\mathbb{P}^4}$. The horizontal arrows on the right side are also isomorphisms by well known properties. If we "contract the isomorphisms" we can write the diagram above more succinctly

$$\begin{array}{ccc}
& & \varphi_{2*} \mathcal{O}_{W_{0,2}}(3) \\
& \nearrow & \downarrow \\
\mathcal{O}_E(-5) & & \tau_* \mathcal{O}_T(3) \\
& \searrow &
\end{array}$$

By push-pull we get a global section $y_2 \in H^0(\mathcal{O}_{W_{0,2}}(3) \otimes \varphi_2^* \mathcal{O}_E(5))$ which restricts on T to the global section $s \in H^0(\mathcal{O}_T(3) \otimes \tau^* \mathcal{O}_E(5))$ defining S inside T , cf. (2.1.b).

We claim that the scheme-theoretic vanishing locus of y_2 is precisely Y_2 . First, the fact that y_2 vanishes on Y_2 can be seen from construction as follows. We have started with the fact that $\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1} \subset \mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^3$, but more can be said: $\mathcal{I}_{Z/\mathbb{P}^4 \times \mathbb{A}^1} \otimes \mathcal{O}_E$ actually lies in the kernel of the map $\mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^3 \otimes \mathcal{O}_E \rightarrow \mathcal{I}_{E/Z}^3 \otimes \mathcal{O}_E$, which is a piece of the map of graded \mathcal{O}_E -algebras

$$\bigoplus_{k \geq 0} \mathcal{I}_{E/\mathbb{P}^4 \times \mathbb{A}^1}^k \otimes \mathcal{O}_E \longrightarrow \bigoplus_{k \geq 0} \mathcal{I}_{E/Z}^k \otimes \mathcal{O}_E$$

corresponding to the closed immersion $Y_2 \rightarrow W_{0,2}$, thus proving that y_2 vanishes on Y_2 . The fact that y_2 is precisely the defining equation of Y_2 follows from the observation that y_2 restricts on T to the section s defining S .

For the purpose of giving explicit equations, it is most convenient to use the description of our sheaves as symmetric powers of normal bundles. Linear algebra gives a canonical decomposition

$$\mathrm{Sym}^3(\mathcal{N} \oplus T_0 \mathbb{A}^1 \otimes \mathcal{O}_E)^\vee = \bigoplus_{k=0}^3 \mathrm{Sym}^k \mathcal{N}^\vee \otimes \langle t^{3-k} \rangle \mathcal{O}_E$$

(we are back to writing $T_0 \mathbb{A}^1 \otimes \mathcal{O}_E$ instead of \mathcal{O}_E), or equivalently

$$\varphi_{2*} \mathcal{O}_{W_{0,2}}(3) = \bigoplus_{k=0}^3 \tau_* \mathcal{O}_T(k) \otimes \langle t^{3-k} \rangle \mathcal{O}_E,$$

which once more can be conveniently rearranged using push-pull as

$$H^0(\mathcal{O}_{W_{0,2}}(3) \otimes \varphi_2^* \mathcal{O}_E(5)) \cong \bigoplus_{k=0}^3 H^0(\mathcal{O}_T(k) \otimes \tau^* \mathcal{O}_E(5)) \otimes \mathbb{C}\langle t^{3-k} \rangle. \quad (2.3)$$

The main point is that, by construction,

$$y_2 = (\tau^* i^* P \otimes t^3, 0, 0, s \otimes 1) \quad (2.4)$$

with the identifications above, where i^*P is the restriction of P to E and $\tau^* i^*P$ is its pullback to T . In conclusion, we have obtained an explicit equation for Y_2 inside the projective bundle $W_{0,2}$.

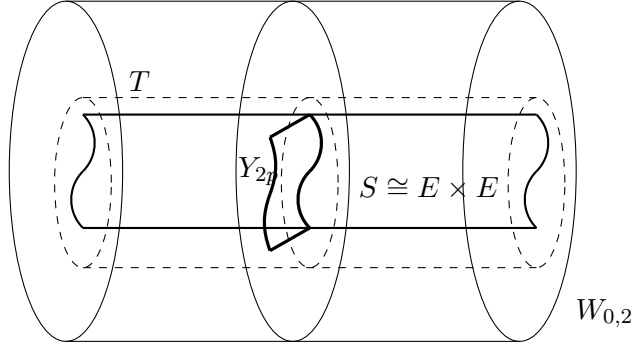


Fig. 2.2. The projective bundle $W_{0,2}$.

Proposition 2.8. *The scheme Y_2 is smooth and irreducible. For closed $p \in E$, the fiber of $\varphi_2 : Y_2 \rightarrow E$ over p is a cubic surface inside the corresponding fiber of $W_{0,2}$ which is either a cone over E with vertex at $\sigma(p)$ if $p \in \{p_1, p_2, \dots, p_{25}\}$ or a special smooth cubic surface which can be expressed as a triple cover of T_p totally ramified at $S_p \cong E \subset T_p$ via projection from $\sigma(p)$ otherwise.*

Proof. Most statements follow immediately from (2.4) since the formula shows that the equation of each fiber of Y_2 inside the corresponding fiber of $W_{0,2}$ is of the form $c_p T^3 + F_3(X, Y, Z) = 0$ for some constant c_p which is zero precisely when $p \in \{p_1, p_2, \dots, p_{25}\}$. The only slightly nontrivial thing that's left to check is the smoothness of Y_2 at the 25 points $\sigma(p_i)$, but this follows easily from the fact that $\partial c / \partial p(p_i) \neq 0$, which in turn follows from the fact that p_1, p_2, \dots, p_{25} are distinct. The derivative above is actually well-defined since sections of line bundles admit derivatives at vanishing points. \square

Proposition 2.9. *For general choices, Y_1 and Y_2 are smooth irreducible divisors on X meeting transversally and X is smooth in a neighborhood of the central fiber.*

Proof. It's worth noticing that since X is a Cartier divisor on W , there will be no need to worry about embedded components in this proof. Smoothness and irreducibility of Y_1 and Y_2 have already been checked.

First, let's verify smoothness of the total space. It suffices to check smoothness at closed $p \in X_0$, say $p \in Y_i$. The idea now is to use once more the embedding in W . If X

were singular at p , then the intersection $Y_i = X \times_W W_i$ would also be singular at p , which is a contradiction. Slightly more precisely, since "fiber products commute with tangent spaces" by functor of points nonsense, $T_p Y_i = T_p X \times_{T_p W} T_p W_i$ and since $T_p Y_i \neq T_p W_i$ then also $T_p X \neq T_p W$ proving smoothness.

Transversality of Y_1 and Y_2 is straightforward. We may argue that their intersection is S purely formally: $Y_1 \times_X Y_2 = Y_1 \times_W W_{0,2} = \tilde{N} \times_{\tilde{P}} T = S$, which is smooth, so transversality follows. \square

2.3. Triple Cyclic Covers of \mathbb{P}^2 . We conclude this section with an elementary remark on the special class of cubic surfaces encountered earlier. If S is a cubic surface whose equation in \mathbb{P}^3 is given in coordinates by

$$F_3(X, Y, Z) + T^3 = 0$$

for some homogeneous degree 3 polynomial $F_3 \in \mathbb{C}[X, Y, Z]$ defining a smooth plane cubic E , then projection from the point $[0 : 0 : 0 : 1]$ to the plane given by $T = 0$ exhibits S as a triple cover totally branched along E and étale elsewhere.

As it is the case with all smooth cubic surfaces, $\text{Pic } S$ is (over-)generated by the divisor classes of the 27 lines. However, the configuration of lines is special: the 27 lines come in 9 triples of coplanar lines passing through each of the 9 flex points of E . Indeed, the plane determined by the center of projection and any of the triple tangents to E cuts S along a plane cubic with a triple point at the corresponding flex of E and it therefore has to be a union of 3 lines. Consequently, the image of the restriction map $\text{Pic } S \rightarrow \text{Pic } E$ consists precisely of the line bundles on E whose cube is some $\mathcal{O}_E(k)$. Finally, we prove some basic existence results which will be used in the proof of the existence of rigid stable maps of arbitrary genus.

Lemma 2.10. Let f_1 and f_2 be two distinct flex points of E and l_1 an arbitrary line in S passing through f_1 . Then there exists a unique line $l_2 \subset S$ passing through f_2 which intersects l_1 .

Proof. We will give an indirect argument proving uniqueness first. If there were two such lines l_2 and l'_2 , then l_1 would have to lie in the plane spanned by l_2 and l'_2 . However, the only other line on S contained in the plane spanned by l_2 and l'_2 is just the third line on S passing through f_2 , which is different from l_1 thus proving uniqueness. Existence can be inferred from the fact that l_1 has to intersect 10 other lines on S . Two of them are the other lines passing through f_1 , so there are 8 more lines to account for. However, there are precisely 8 flex points of E besides f_1 , so there can only be precisely one line intersecting l_1 through each such point. \square

Lemma 2.11. Let $k \leq 3$ and D_E be an effective divisor on E such that $\mathcal{O}_E(3D_E) \cong \mathcal{O}_E(k)$. Then there exists an effective divisor D_S on S such that $D_S \cap E = D_E$ and the complete linear system $|D_S|$ is either: (1) a singleton consisting of a line if $k = 1$; (2) a pencil of conics if $k = 2$; (3) either (3a) a net of twisted cubics or (3b) the anticanonical linear system, if $k = 3$. Moreover, in (3b), D_S can be chosen to be singular.

Proof. If $k = 1$, we may just pick any of the 3 lines through $D_E = p$. If $k = 2$, there exist two distinct flex points of E p and q such that $D_E \sim p + q$. Let l_p be any line

through p . By the previous lemma, there exists a line l_q through q which intersects l_p . Then $|l_p + l_q|$ is a pencil of conics and it is easy to see that the restriction $|l_p + l_q| \rightarrow |D_E|$ is bijective, so there exists $D_S \in |l_p + l_q|$ such that $D_S \cap E = D_E$.

Finally, if $k = 3$, there exist 3 distinct flex points of E p, q, r such that $D_E \sim p + q + r$. Let l_p be a line through p and l_q, l_r the lines passing through q and r which intersect l_p . Then there are two cases:

(3a) $l_q \cap l_r = \emptyset$. In this case, $|l_p + l_q + l_r|$ is a net of twisted cubics and the restriction $|l_p + l_q + l_r| \rightarrow |D_E|$ is again bijective, so there exists $D_S \in |l_p + l_q + l_r|$ such that $D_S \cap E = D_E$.

(3b) $l_q \cap l_r \neq \emptyset$. Now $|l_p + l_q + l_r|$ is the anticanonical linear system, i.e. the linear system of hyperplane sections. In this case, the (now rational) restriction map $|l_p + l_q + l_r| \rightarrow |D_E|$ is surjective with fibers of dimension one. The fiber of D_E is a pencil of hyperplane sections, so we can pick D_S to be a singular member of this pencil. \square

3. DEGENERATE STABLE MAPS TO THE CENTRAL FIBER

As emphasized in the introduction, the motivation for the choice of the specific degeneration worked out in the previous section is the presence of the two nonconstant maps to E , which greatly restricts the position of the limits of the nearby rational curves and, to a lower extent, of nearby curves of arbitrary genus. The theoretical framework for working with stable maps to degenerations has been laid out in [Li01, Li02]. The degeneration formula for Gromov-Witten invariants proved in loc. cit. will be less important to us than the construction of the moduli spaces, namely the extension of the family of moduli spaces of stable maps over the singular fiber. Very roughly, this degenerate moduli space parametrizes *locally* deformable (to nearby fibers) maps to X_0 with semistable source - with a twist, namely the idea to allow an "expansion" of X_0 by inserting a chain of ruled varieties between Y_1 and Y_2 with the purpose of avoiding contracted components (of the source curve) to the singular locus of X_0 .

Recall [Li01] that to the total space X and the pair $Y_i^{\text{rel}} = (Y_i, S)$ we may associate the (Artin) stack of expanded degenerations \mathfrak{X} respectively the stack of expanded pairs $\mathfrak{Y}_i^{\text{rel}}$. The stacks of expanded degenerations and expanded pairs allow one to define the spaces of relative stable maps and stable maps to expanded degenerations, which will be denoted by $M(\mathfrak{X}, \Gamma)$ and $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i)$ respectively. The topological data Γ consists simply of the triple $(g, b, k = 0)$, where g is the arithmetic genus of the source, d is the degree of the stable maps relative to some predetermined relatively ample line bundle H on X and k is the number of ordinary marked points. The discrete data specified in the Γ_i is the following:

- a decorated graph with vertices $V(\Gamma_i)$ corresponding to the connected components of the stable maps, roots $R(\Gamma_i)$ corresponding to the distinguished marked points q_1, \dots, q_r attached to the suitable vertices and no edges;
- a function $\mu : R(\Gamma_i) \rightarrow \mathbb{Z}^+$ assigning the suitable order of intersection with S at each distinguished marked point;
- functions $\beta_i : V(\Gamma_i) \rightarrow H_2(Y_i, \mathbb{Z})$ and $g_i : V(\Gamma_i) \rightarrow \mathbb{N}$ prescribing the homology class in Y_i , respectively the arithmetic genus of each connected component.

The total space forms a family $\pi^M : M(\mathfrak{X}, \Gamma) \rightarrow \Delta = \mathbb{A}^1$ which is [Li01] separated and proper over Δ .

There exist distinguished evaluation morphisms $\mathbf{q}_i : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i) \rightarrow S^r$ (the distinguished marked points get artificially ordered as part of the topological data Γ_i) for $i = 1, 2$. Whenever Γ_1 and Γ_2 are compatible in the sense of having the same number r of roots and the corresponding roots are weighted identically, their fiber product admits a morphism

$$\Phi_\Gamma : M(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1) \times_{S^r} M(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2) \longrightarrow M(\mathfrak{Y}_1^{\text{rel}} \sqcup \mathfrak{Y}_2^{\text{rel}}, \Gamma_1 \sqcup \Gamma_2) \quad (3.1)$$

to a closed substack of $M(\mathfrak{X}_0, \Gamma)$, which glues two relative stable maps along their distinguished marked points. For each compatible pair $\eta = (\Gamma_1, \Gamma_2)$, there exist line bundles \mathbf{L}_η on the total space $M(\mathfrak{X}, \Gamma)$ with global sections \mathbf{s}_η such that the vanishing locus of \mathbf{s}_η is a closed substack $M(\mathfrak{X}_0, \eta)$ of $M(\mathfrak{X}_0, \Gamma)$ which is topologically identical to $M(\mathfrak{Y}_1^{\text{rel}} \sqcup \mathfrak{Y}_2^{\text{rel}}, \Gamma_1 \sqcup \Gamma_2)$.

Having briefly recalled the main objects, we can return to the specifics of the degeneration described in the previous section. For notational purposes, it is useful to introduce the usual space of stable maps $\overline{M}_{0,0}(X, d)$ and the Chow variety $\text{Chow}_d(X)$. First, we have a composition $M(\mathfrak{X}, \Gamma) \rightarrow \overline{M}_0(X, d) \rightarrow \text{Chow}_d(X)$, which we denote by $\gamma_{\mathfrak{X}}$. There is a similar sequence of morphisms for each Y_i

$$M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i) \longrightarrow \prod_{v \in V(\Gamma_i)} \overline{M}_0(Y_i, \beta(v)) \longrightarrow \prod_{v \in V(\Gamma_i)} \text{Chow}_{\beta(v)}(Y_i), \quad (3.2)$$

where v ranges over all vertices of the graph specified by Γ_i and $\beta : V(\Gamma_i) \rightarrow H_2(Y_i, \mathbb{Z})$ gives the class in Y_i of the image of each component.

The expectation that $M(\mathfrak{X}_0, \Gamma)$ has dimension zero is unrealistic because of the presence of multiple covers. However, we may ask whether its image under $\gamma_{\mathfrak{X}}$ has dimension zero (this is false in general as well). Then three questions arise:

- (Q1) Are the images of the (closed) fibers of \mathbf{q}_i under $\gamma_{\mathfrak{X}}$ finite?
- (Q2) Do the images of \mathbf{q}_i have the expected dimensions?
- (Q3) Do these images intersect dimensionally transversally for $i = 1, 2$?

The answer to (Q1) is affirmative. The answer to (Q2) is also morally affirmative. It seems possible that under exceptional circumstances the dimensions of the images of \mathbf{q}_i drop below the expected ones, but this is of no concern. We will prove that they never exceed the expected dimension. Unfortunately, the answer to the third question is in general negative. Nevertheless, the third point turns out to be of purely combinatorial nature and as such, it is possible to determine algorithmically when it does hold.

3.1. Relative Stable Maps to Y_1 and Y_2 . In this section we will describe the moduli spaces $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i)$ of relative stable maps to the \mathfrak{Y}_i , when $g_i \equiv 0$, i.e. all connected components have arithmetic genus 0. Since there are no nonconstant maps from a rational curve to a curve of geometric genus 1, all irreducible components of any relative stable map are sent either inside the fibers of the maps $\varphi_i : Y_i \rightarrow E_i$, or to the lines in the rulings of the intermediary ruled varieties reviewed earlier, possibly as multiple covers. Formally, we have a morphism

$$\varphi_i^M : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i) \longrightarrow E_i^{V(\Gamma_i)} \quad (3.3)$$

specifying the φ_i -fiber containing each connected component. We will show that the conditions the r -tuples of images in S of the distinguished marked points need to satisfy

boil down to a collection of linear equations in $\text{Pic}(E)$. These conditions are obtained by restricting information about rational equivalence on the surfaces $\varphi_i^{-1}(p)$ to information about rational equivalence on the corresponding copy of $E_j \subset \varphi_i^{-1}(p)$.

First, we will address (Q1) above. To avoid mentioning the awkward maps to Chow varieties at each step, we introduce some ad hoc terminology. Let $\pi : C \rightarrow S$ be a flat family of semistable curves over a scheme (or perhaps DM-stack) and $f : C \rightarrow X$ a morphism to some projective variety X , inducing a morphism $S \rightarrow \text{Chow}(X)$. Let $g : S \rightarrow Y$ be a proper morphism to a (complex) scheme Y . We say that g is *cycle-finite* relative to the family of maps $C \rightarrow X$ if the image in $\text{Chow}(X)$ of any closed fiber $g^{-1}(y)$ is zero-dimensional. In this language question (Q1) becomes: are the distinguished evaluation morphisms \mathbf{q}_i cycle-finite? We prove that this is the case.

Proposition 3.1. *The distinguished evaluation morphisms \mathbf{q}_i are cycle-finite relative to the universal family over $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i)$ mapping into Y_i .*

Proof. Roughly, positive-dimensional families would be forced to split off a component in S , which is impossible since S doesn't contain rational curves.

Denote the universal family by $\pi_1 : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i + 1) \rightarrow M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i)$. The notation $\Gamma_i + 1$ denotes a topological type identical to Γ_i , with the exception of the existence of a single ordinary marked point. Let

$$\text{ev}_1 : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i + 1) \longrightarrow Y_i$$

be the evaluation morphism associated to this point. Let $s = (s_1, \dots, s_r) \in S^r$ be a closed point. Our claim is essentially that the image Σ of $\pi_1^{-1}(\mathbf{q}_i^{-1}(s))$ under ev_1 in Y_i has dimension one. The crucial observation is this: since $S = E \times E$ contains no rational curves, all components of the relative stable maps which map to the intermediary ruled threefolds over S will be mapped as covers of some fibers of these rulings; this implies that $\Sigma \cap S$ is supported on $\{s_1, \dots, s_r\}$, so Σ can't have dimension 2 or more. \square

There are some basic facts that need to be clarified before moving on to (Q2). We choose once and for all an arbitrary flex point of $E \subset \mathbb{P}^4$ to be the zero element of $(E, +)$. This gives natural identifications of all connected components of $\text{Pic}(E)$ with E and in particular, a preferred isomorphism $E_1 \cong E$. Consider the pushforward maps $\varphi_{i*} : H_2(Y_i, \mathbb{Z}) \rightarrow H_2(E_i, \mathbb{Z}) \cong \mathbb{Z}$. Let $K_1 \cong \mathbb{Z}[\text{fiber}] \oplus \mathbb{Z}[\text{directrix}] \subset H_2(Y_1, \mathbb{Z})$ generated by the classes of the directrix respectively any line in the ruling of any fiber of φ_1 and K_2 generated by the classes of all lines on any smooth fiber of φ_2 . Then $\text{Im}(\beta_i) \subset K_i$, for $i = 1, 2$.

For any $v \in V(\Gamma_i)$, we denote by $\Gamma_i|_v$ the restriction of the combinatorial data to the vertex v , i.e. the collection of data consisting of the graph with the unique vertex v , the roots which were previously attached to v , the class $\beta_i(v)$, the genus function $g_i(v) = 0$ and no legs, corresponding to no ordinary marked points. Consider the moduli stack $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v)$ of relative stable maps of this topological type. Again, there is a morphism

$$\varphi_{i|v}^M : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v) \longrightarrow E_i.$$

Let $\Gamma_i|_v + 1$ denote the topological type which is identical to $\Gamma_i|_v$, with the exception of the existence of one ordinary marked point. We obtain the universal family $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v + 1)$ over the previous moduli space. We have a (2-)commutative diagram

$$\begin{array}{ccc}
M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v + 1) & \longrightarrow & Y_i \\
\downarrow & & \downarrow \\
M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v) & \longrightarrow & E_i
\end{array}$$

inducing a map $M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v + 1) \rightarrow M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v) \times_{E_i} Y_i$. Define the maps

$$\omega_i : M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v) \longrightarrow \text{Pic}^{\sum_{\alpha \in R(v)} \mu(\alpha)}(E_j)$$

which associate to each relative stable map $(C, f, (q_\alpha)_{\alpha \in R(v)})$ the divisor class cut out by $f(C)$ on $\{\varphi_{i|v}^M(f)\} \times E_j$, i.e. $\mathcal{O}_{E_j}(\sum_{\alpha \in R(v)} \mu(\alpha) f(q_\alpha))$. The degree $\sum_{\alpha \in R(v)} \mu(\alpha)$ will be denoted by $\sigma(v)$. The first step is to understand the image of the map $\omega_i \times \varphi_{i|v}^M$ in $\text{Pic}^{\sigma(v)}(E_j) \times E_i$. Identify the target with $E_j \times E_i = E_1 \times E_2$ and let (x, y) be coordinates on $E_1 \times E_2$.

Lemma 3.2. (a) Let $i = 1$ and $\beta_1(v) = k_f[\text{line}] + k_\infty[\text{directrix}] \in K_1$. Then the image is contained inside the locus in $E_1 \times E_2$ given by the equation $y - (k_f - 2k_\infty)x = 0$.

(b) Let $i = 2$ and $\beta_2(v) \in K_2$ of degree k_l relative to $\mathcal{O}_{W_{0,2}}(1)$. Then the image is contained in the union of the curve of equation $3x = k_ly$ with $E_1 \times \{p_1, \dots, p_{25}\}$.

Proof. (a) This follows from part (a) of Claim 2.4. Indeed, for $\mathcal{D} \in \text{Pic}^2(E) = E_1$, 2.4-(a) says that the divisor class $\mathcal{O}_\Sigma(f)^{\otimes k_f} \otimes \mathcal{O}_\Sigma(e_\infty)^{\otimes k_\infty}$ on the fiber $\Sigma = \varphi_1^{-1}(\mathcal{D})$ restricts on the corresponding copy of $E_2 = E$ sitting inside Σ to $\mathcal{O}_{\mathbb{P}^4}(k_\infty)|_E \otimes \mathcal{D}^{\otimes(k_f - 2k_\infty)}$. Given that the implicit isomorphism of the corresponding connected component of $\text{Pic}(E)$ with E is induced by a hyperflex point, the first terms disappears, so the image lies inside the locus $y = (k_f - 2k_\infty)x$.

(b) Similarly, this follows from part (b) of the same claim and the discussion in section 2.3. Let $p \in E_2 = E$. Assume that $p \notin \{p_1, p_2, \dots, p_{25}\}$. Then the fiber $\Sigma = \varphi_2^{-1}(p)$ of φ_2 is a smooth cubic surface, as in section 2.3. By the discussion in that section, any divisor class on Σ which pushes forward to $k_l[\text{line}]$ in $H_2(Y_2, \mathbb{Z})$ has the property that its cube is the class $\mathcal{O}_{\mathbb{P}^4}(k_l)|_E \otimes \mathcal{O}_E(-2k_lp)$, so, with the implicit identifications, we get the desired constrain $3(x - k_ly) + 2k_ly = 0$. The reason for the term $x - k_ly$ is the following: the coordinate x on $E_1 \times E_2$ is actually the coordinate μ on the canonical $E \times E$ and the corrections will stack in $\text{Pic}^{k_l}(E_1)$. \square

Let $I(\Gamma_i|_v)$ be the loci in $\text{Pic}^{\sigma(v)}(E_j) \times E_i$ described above. Consider the diagram

$$\begin{array}{ccc}
& & (E_1 \times E_2)^{R(v)} \\
& \nearrow \mathbf{q}_{i|v} & \uparrow \text{Id}_{E_j^{R(v)}} \times \text{Diag}_{E_i} \\
M(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i|_v) & \longrightarrow & E_j^{R(v)} \times E_i \\
& \searrow \omega_i \times \varphi_{i|v}^M & \downarrow h \\
& & \text{Pic}^{\sigma(v)}(E_j) \times E_i
\end{array}$$

where $\mathbf{q}_{i|v}$ is the evaluation morphism restricted to v and h is defined fiberwise by

$$h((y_\alpha)_{\alpha \in R(v)}, x) := \left(\mathcal{O}_{E_j} \left(\sum_{\alpha \in R(v)} \mu(\alpha) y_\alpha \right), x \right)$$

Define $A(\Gamma_{i|v})$ to be the image of $h^{-1}(I(\Gamma_{i|v}))$ in $(E_1 \times E_2)^{R(v)}$. Then $A(\Gamma_{i|v})$ is an equidimensional union of abelian subvarieties of $(E_1 \times E_2)^{R(v)}$ of dimension

$$\dim A(\Gamma_{i|v}) = \dim(E_j^{R(v)} \times E_i) - 1 = |R(v)|$$

and the image of $\mathbf{q}_{i|v}$ is contained in $A(\Gamma_{i|v})$. Let $A(\Gamma_i) \subset (E_1 \times E_2)^r$ be the direct product of all $A(\Gamma_{i|v})$ over all $v \in V(\Gamma_i)$. Then the image of \mathbf{q}_i is contained in $A(\Gamma_i)$, which is a union of abelian subvarieties of $(E_1 \times E_2)^r$ of pure dimension $\sum_{v \in V(\Gamma_i)} |R(v)| = r$. Note that $A(\Gamma_1)$ and $A(\Gamma_2)$ have complementary dimensions in S^r , as we should expect.

Remark 3.3. Given any abelian variety and any two abelian subvarieties of complementary dimensions, either the two abelian subvarieties intersect transversally, or the intersection of the associated homology classes is zero. This can be seen, for instance, from the excess intersection formula.

From the point of view of the degeneration formula proved by Li, this seems to have intriguing consequences: any potentially infinite⁵ family of limits is necessarily associated with the vanishing of the corresponding part of the contribution to the virtual count. However, a huge technical difficulty is the following: for $i = 2$ the image consists not of an abelian subvariety, but a singular union of subvarieties and it is entirely unclear whether it is possible to separate the contributions from each individual component in a geometrically meaningful way.

3.2. Combinatorics of the Degenerate Maps. In this section we write down explicitly the combinatorial laws governing the stable maps to \mathfrak{X}_0 . Let $\eta = (\Gamma_1, \Gamma_2)$ be a pair of compatible topological types, i.e. Γ_1 and Γ_2 have the same number r of distinguished marked points and the identically indexed distinguished marked points have the same μ -weight. As we said, we will only be concerned with the case $g_1 \equiv g_2 \equiv 0$, which covers all possibilities in the case $g = 0$ respectively some of the possibilities when $g > 0$.

To avoid notation becoming unnecessarily cryptic, we will work with only one map instead of families. Fix a single stable map $(C, f, q_1, \dots, q_r) \in M(\mathfrak{X}_0, \eta)$. Let G be the graph with vertices $V(G) = V(\Gamma_1) \cup V(\Gamma_2)$ and r edges obtained by glueing the roots with identical indexing. Then $\dim H^1(G) = g$ since

$$g = \dim H^1(G) + \sum_{v \in V(\Gamma_1)} g_1(v) + \sum_{v \in V(\Gamma_2)} g_2(v)$$

and we are assuming $g_1 \equiv g_2 \equiv 0$. From now on, we will call the elements of $V(\Gamma_1)$ red vertices and the elements of $V(\Gamma_2)$ green vertices. As in lemma 2.3, let $k_f, k_\infty : V(\Gamma_1) \rightarrow$

⁵By infinite we mean sweeping out at least a two-dimensional locus on X_0 , so e.g. multiple covers of a fixed curve don't count.

\mathbb{N} such that $\beta_1 \equiv k_f[\text{line}] + k_\infty[\text{directrix}]$ and $k_l : V(\Gamma_2) \rightarrow \mathbb{N}$ such that β_2 has degree k_l relative to $\mathcal{O}_{W_{0,2}}(1)$.

Let $\lambda : V(G) \rightarrow (E, +) \cong (\mathbb{R}^2/\mathbb{Z}^2, +)$ be the map giving the components of $\varphi_1^M([f])$ and $\varphi_2^M([f])$; we may merge $i = 1$ and $i = 2$ since we are simultaneously identifying E_1 and E_2 with E . The point is that lemma 3.2 and the discussion thereafter imposes $|V(G)|$ linear conditions on λ , which are usually linearly independent. First, let v be a red vertex. Then the condition imposed on λ is

$$\sum_{w \in V(v)} \mu([vw])\lambda(w) = (k_f(v) - 2k_\infty(v))\lambda(v)$$

with the numerical constrain on the multiplicities

$$\sum_{w \in V(v)} \mu[vw] = 2k_f(v) + k_\infty(v),$$

since $(k_f[\text{line}] + k_\infty[\text{directrix}] \cdot -K_{\mathbb{P}^1})_{\mathbb{P}^1} = 2k_1^f(v) + k_1^\infty(v)$ by a straightforward calculation. We are denoting the set of neighbors of v in G by $V(v)$.

Now let v be a green vertex. We have two cases: either $\lambda(v) \in \{p_1, p_2, \dots, p_{25}\}$ in which nothing more can be said, or otherwise, again from the computation in 3.2 and the following discussion, we get

$$3 \sum_{w \in V(v)} \mu([vw])\lambda(w) = k_l(v)\lambda(v)$$

and the numerical constrain on the weights

$$\sum_{w \in V(v)} \mu([vw]) = k_l(v)$$

Note that the latter constrain still holds in the case $\lambda(v) \in \{p_1, p_2, \dots, p_{25}\}$. We introduce some (fairly standard) notation to state the equations above in a slightly more pleasant form.

Definition 3.4. Let G be any simple connected graph without isolated vertices and let $\mu : E(G) \rightarrow \mathbb{Z}^+$ be a weight function on the set of edges. First, we naturally define the weighted degrees of the vertices as

$$\deg_\mu(v) = \sum_{w \in V(v)} \mu([vw]).$$

For any function f defined on the set of vertices $V(G)$ with values in some abelian group, we define the μ -weighted unnormalized G -Laplacian of f by

$$\Delta_\mu^u f(v) := \sum_{w \in V(v)} \mu([vw])(f(v) - f(w))$$

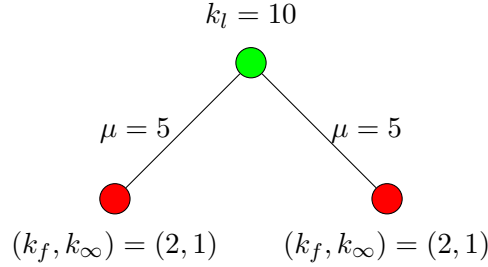
The normalized μ -weighted G -Laplacian Δ_μ is defined by the same formula, but dividing by $\deg_\mu(v)$, assuming that $\deg_\mu(v)$ is invertible in the target group. If $\mu \equiv 1$, we suppress this subscript.

A basic computation allows us to rewrite the conditions above as follows:

$$\begin{cases} \Delta_\mu^u \lambda(v) = (3 \deg_\mu(v) - 5k_f(v)) \lambda(v) & \text{if } v \text{ is red,} \\ 3\Delta_\mu^u \lambda(v) = 2 \deg_\mu(v) \lambda(v) \text{ or } \lambda(v) \in \{p_1, p_2, \dots, p_{25}\} & \text{if } v \text{ is green.} \end{cases} \quad (3.4)$$

In conclusion, (cycle-)finiteness of $M(\mathfrak{X}_0, \eta)$ boils down to the system of linear equations in $(E, +)$ above having only finitely many solution. The smallest counterexample to finiteness occurs (for $g = 0$) in degree $d = 6$.

Example 3.5.⁶ As soon as $r = 2$ we may find examples in which the system (3.4) is special. Let G have two red vertices R_1, R_2 and a green vertex G of "smooth" type. Let $k_f(R_i) = 2$, $k_\infty(R_i) = 1$, $k_l(G) = 10$ and let both edges have weight 5.



An immediate calculation shows that (3.4) is special in this case. Geometrically, the red vertices correspond to twisted cubics which are sections of the corresponding \mathbb{F}_1 surfaces by the same osculating hyperplane in \mathbb{P}^4 to a hyperflex of E .

3.3. Existence of Rigid Stable Maps. In this section, we prove theorem 1.3 by explicitly exhibiting rigid degenerate stable maps to \mathfrak{X}_0 , of arithmetic genus g and degree $d \gg g$ with certain additional properties and invoking the existence of a perfect obstruction theory on $M(\mathfrak{X}, \Gamma)$ to infer that they are indeed limits of rigid stable maps to nearby fibers of $X \rightarrow \mathbb{A}^1$. Smoothness of the source follows easily from the smoothness of the connected components of the maps to $\mathfrak{Y}_1^{\text{rel}}$ and $\mathfrak{Y}_2^{\text{rel}}$.

The chaotic nature of the combinatorial problem encountered earlier works to our advantage, in that we have an enormous amount of freedom in choosing the topological types. The first main assumption is $\beta_1 \equiv [\text{line}]$, that is, we are assuming that all red vertices correspond to fibers of the rulings in the corresponding Hirzebruch surfaces. Furthermore, we assume $\mu \equiv 1$, hence all red vertices have degree 2. Then the first branch of equation (3.4) simply reads $\lambda(v) = \lambda(w_1) + \lambda(w_2)$, where v is any red vertex and w_1, w_2 are its two neighbors.

Let G' be the graph obtained from G by suppressing all red vertices and replacing any length 2 chain in G between two green vertices with an edge. Then the second branch of equation (3.4) becomes simply $3\Delta^u \lambda(v) = 5 \deg(v) \lambda(v)$, where the discrete Laplacian is now taken relative to G' . We further assume that $\deg v \leq 3$ for all vertices of G' and proceed to construct the degenerate curve.

Claim 3.6. For $d, g \geq 1$, $d \gg g$, there exists a set of data as follows:

⁶This example was found by Gabriel Bujokas

- A graph simple connected graph $G = (V, E)$ with $|E| = d$ and $h^1(G) = g$, hence $|V| = d - g + 1$. We require that $\deg v \leq 3$ for all $v \in V$.
- If \mathbb{K} is some field of characteristic $\neq 2, 3$, consider the discrete equation

$$\Delta\lambda = \frac{5}{3}\lambda \quad (3.5)$$

in $\text{Fun}(V, \mathbb{K})$. We require that (3.5) has only the trivial solution $\lambda = 0$ when $\mathbb{K} = \mathbb{R}$, i.e. $5/3$ is not an eigenvalue of the Laplacian.

- A prime number $p \geq 7$ and a solution $\hat{\lambda}$ of (3.5) for $\mathbb{K} = \mathbb{F}_p$ such that:
 1. $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $1 \leq \text{dist}(v, w) \leq 2$; and
 2. $\hat{\lambda}$ is a *strongly irreducible* solution of (3.5), in the following sense. For any induced subgraph $H \subset G$, consider the similar discrete differential equation

$$F_H\lambda := \Delta_H\lambda - \frac{5}{3}\lambda = 0.$$

Then we require that $\hat{\lambda}$ has the following property: if $F_H\hat{\lambda}(v) = 0$, then $\deg_H v$ is equal to either $\deg_G v$ or 0.

Lemma 3.7. If no two degree 3 vertices of G are adjacent, then (3.5) only has the trivial solution 0 for $\mathbb{K} = \mathbb{R}$ or equivalently \mathbb{Q} .

Proof. Let $\mathbb{K} = \mathbb{Q}$ and let $v_3 : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{+\infty\}$ be the valuation at the prime 3. Pick any solution λ of (3.5). Then we have

$$v_3(\lambda(v)) = v_3\left(\sum_{w \in V(v)} \lambda(w)\right) - v_3(\deg v) + 1 \geq \min_{w \in V(v)} v_3(\lambda(w)) + 1 - v_3(\deg v).$$

Hence any vertex v has a neighbor with strictly smaller $v_3 \circ \lambda$, if $\deg v \leq 2$ and $\lambda(v) \neq 0$, respectively no greater $v_3 \circ \lambda$, if $\deg v = 3$. This easily implies $v_3 \circ \lambda \equiv +\infty$, so $\lambda \equiv 0$. \square

Claim 3.8. There exists a prime $p \geq 7$ and three graphs G_1, G_2, G_3 with eigenvectors $\hat{\lambda}_i$ in \mathbb{F}_p satisfying the conditions in 3.6 and 3.7 (d and g are not fixed) and two elements $a, b \in \mathbb{F}_p$ such that:

- $\#E(G_1)$ and $\#E(G_2)$ are coprime;
- $h^1(G_1) = h^1(G_2) = 1$ and $h^1(G_3) = 2$;
- all graphs G_i contain an edge $[vw]$ such that v and w both have degree 2 in G_i and $\hat{\lambda}_i(v) = a, \hat{\lambda}_i(w) = b$.

Proof. The proof is by explicit example. We defer it to the appendix. \square

Proof of 3.6. We will use the graphs G_1, G_2, G_3 constructed in 3.8 as the building blocks for constructing G . For each G_i , we may define a graph H_i with $|E(H_i)| = |E(G_i)| + 1$ and $h^1(H_i) = h^1(G_i) - 1$ obtained by duplicating the edge $[vw]$ and then separating the two ends, i.e. H_i will begin and end with a copy $[vw]$, like in our pictorial representation of the G_i .

Starting with some G_i , we may insert copies of various H_j 's at the edges which are copies of $[vw]$. Note that we may glue the functions from vertices to \mathbb{F}_p and crucially,

the property of being eigenvectors with eigenvalue $5/3$ is preserved since any collections of vertices consisting of some arbitrary vertex and all its neighbors is contained isomorphically in some G_i . It is also not hard to see that the conditions $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $1 \leq \text{dist}(v, w) \leq 2$ respectively $\deg v \neq 3$ if v has some neighbor of degree 3 are preserved as well through this process. To ensure $h^1(G) = g$, it is enough to insert H_3 precisely $g - 1$ times throughout the process. Finally, since $\#E(G_1)$ and $\#E(G_2)$ are coprime, the elementary "coin problem" ensures that G may have any sufficiently high number of edges. \square

Lemma 3.9. For $d \gg g$, $g \geq 1$ there exists a stable map $f_0 : C_0 \rightarrow X_0$, $[f_0] \in M(\mathfrak{X}_0, \Gamma)$ obtained by glueing two relative stable maps $f_1 \in M(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1)$ and $f_2 \in M(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2)$ mapping to $Y_1[0]$ respectively $Y_2[0]$ such that all connected components of the source of f_i are smooth and are mapped generically one-to-one onto their images and $[f_0]$ is an isolated point of $M(\mathfrak{X}_0, \Gamma)$.

Proof. We will avoid going into the details of this proof, since the desired degenerate map is obtained simply by reversing the discussion so far, with few significant new ingredients. First, we use the graphs constructed in 3.6. as the G' in the discussion at the beginning of this section. Using these, we may reconstruct G , the dual graph of C . Second, we use the solution $\hat{\lambda}$ of (3.5) in \mathbb{F}_p to reconstruct the desired solution λ of (3.4) as follows: first, we extend $\hat{\lambda}$ to $(E, +)$ by first embedding $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ diagonally into $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, which naturally sits inside $(\mathbb{R}^2/\mathbb{Z}^2, +) \cong (E, +)$, then define λ on the red vertices simply by $\lambda(v) = \lambda(w_1) + \lambda(w_2)$, where w_1, w_2 are the two green neighbors of some red v .

Next, we construct the components $f_v : C_v \rightarrow Y_i$ of the stable map. We start with v green, i.e. $i = 2$. Consider the divisor $D = \sum_{w \in V(v)} \lambda(w)$ on the copy of E sitting inside the special cubic surface $\Sigma = \varphi_2^{-1}(\lambda(v))$. The fact that λ is a solution of (3.4) implies that we are in the situation of lemma 2.11. Let D_Σ be the divisor on Σ such that $D_\Sigma \cap E = D$, as constructed in the said lemma. The crucial point (which will only be sketched) is that D_Σ is irreducible. This is where the (combinatorial) strong irreducibility condition comes into play. Indeed, if the divisor were reducible, then either component would intersect the boundary E at points which satisfy the analogous linear conditions in $(E, +)$ - precisely what is ruled out by the strong irreducibility assumption. We are implicitly relying on the equivalence of the linear conditions for $\hat{\lambda}$ and for λ . In conclusion, we can define $f_v : C_v \cong \mathbb{P}^1 \rightarrow Y_2$ to be the normalization of the divisor above. Finally, the distinguished marked points on C_v are the preimages of E under f_v . The fact that they are distinct is ensured by the condition $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $\text{dist}(v, w) = 2$ in G' .

The case v red is similar but less laborious. Again, the fact that λ is a solution of (3.4) ensures that we may find such a line. The property that the distinguished marked points are distinct is ensured by $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if v and w are neighbors in G' .

All in all, we may glue f_1 and f_2 to obtain a morphism $f_0 : C_0 \rightarrow X_0$, where C_0 is obtained by glueing nodally all components of C_v along the pair of identically indexed distinguished marked points. Note that the fact that the distinguished marked points were obtained distinct directly from the construction above means that the target of f_0 is indeed X_0 rather than some higher $X_0[n]$. It is clear that $\text{Aut}(f_0)$ is trivial, so f_0 is stable. The crucial condition that $[f_0]$ is isolated in $M(\mathfrak{X}_0, \Gamma)$ follows from the second

bullet in 3.6. \square

Proof of 1.3. If $g = 0$, the statement is known by work of Katz, as explained in the introduction. If $g \geq 1$, the previous lemma all but completes the proof of the theorem. Since $M(\mathfrak{X}, \Gamma)$ admits a perfect obstruction theory of the expected dimension [Li02], the irreducible component of $M(\mathfrak{X}, \Gamma)$ containing the point $[f_0]$ must have dimension at least one. In fact, since the intersection of this locus with the central fiber has an isolated point at $[f_0]$, it must be of dimension precisely 1 and, moreover, it must not be contained in the central fiber. It follows that we can find an \mathbb{A}^1 -morphism $h : B \rightarrow M(\mathfrak{X}, \Gamma)$ from a smooth affine curve B with a distinguished point $b_0 \in B$ admitting a map $\psi : B \rightarrow \mathbb{A}^1$ such that $h(b_0) = [f_0]$ lies in the central fiber, but no other $h(b)$, $b \in B$ does. We may choose B such that all $h(b_t)$ are isolated points of $\overline{M}_{0,0}(X_{\psi(t)}, d)$.

Finally, we have to argue that the source C_t of $h(b_t) = [(C_t, f_t)]$ is smooth for all t in a punctured neighborhood of $b_0 \in B$, where $C \rightarrow B$ is the pullback to B of the universal family of stable maps. Consider the commutative diagram

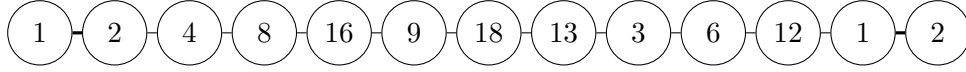
$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\psi} & \mathbb{A}^1 \end{array}$$

inducing a map $C \rightarrow X \times_{\mathbb{A}^1} B$. Any singular point of C_0 is a distinguished marked point q , so, in particular it is a node. The versal deformation space of a node is the germ at the origin of the family $\text{Spec } \mathbb{C}[x, y, s]/(s - xy) \rightarrow \text{Spec } \mathbb{C}[s]$, so it suffices to prove that C is locally irreducible near q in the complex-analytic topology. If it wasn't, then it would have two components Z_1 and Z_2 , indexed such that Z_i contains the branch of f_0 near q mapping to Y_i . Then the pullback of the line bundle $\mathcal{O}_X(Y_1)$ with the section $1 \in H^0(\mathcal{O}_X(Y_1))$ restricts on Z_2 to a line bundle with a section vanishing only at q , which is impossible. \square

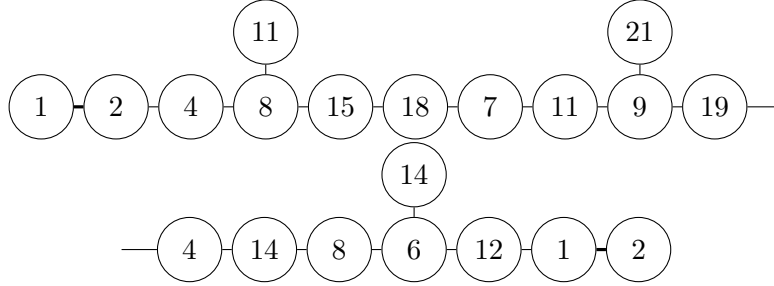
Remark 3.10. The sufficient bound we have obtained is $d \geq 43g + 170$. Of course, this depends on the explicit examples exhibited in the appendix. It would be interesting to know asymptotically what the optimal bound is.

4. APPENDIX: PROOF OF 3.8 BY EXAMPLE

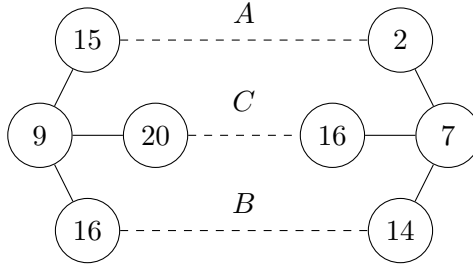
We have deferred the proof by example of Claim 3.8 to this appendix. The examples below were found using a computer program in C++. In principle, they can be checked by hand, but I have written a second (much simpler) computer program available on my personal website, only for the purpose of verifying these examples. The prime number is $p = 23$. Below, we draw G_1, G_2, G_3 and write the corresponding value of $\hat{\lambda}_i$ inside each vertex. For formatting reasons, G_1 and G_2 are actually drawn as trees; the actual G_1 and G_2 are obtained by glueing the distinguished $[1 - 2]$ edges. Note that G_1, G_2, G_3 have 11, 18 respectively 43 edges. This is G_1 :



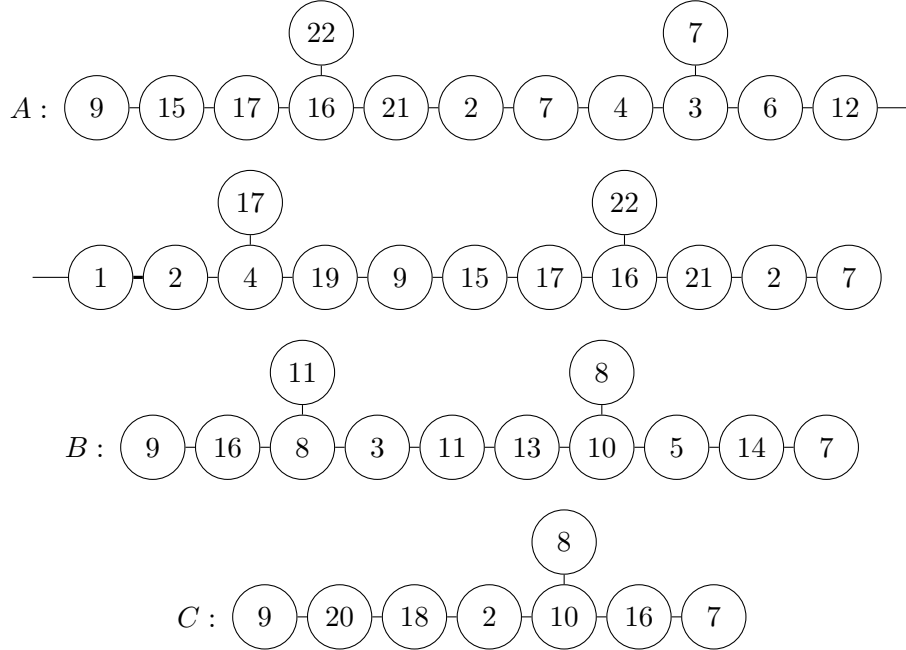
This is G_2 (glue the two components as suggested pictorially):



Finally, this is G_3 :



obtained by completing with the branches A, B, C given below.



Insert the the 3 branches A, B, C to the previous diagram as indicated to obtain G_3 .

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